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A COMPLETE RE-DERIVATION OF THE HIDDEN VARIABLE EQUATION IN DAVID BOHM'S 1952 PAPERS

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ABSTRACT

Schrodinger's equation is a basic mathematical expression for a single particle in quantum mechanics. However, its physical meaning has never been assigned in any standard quantum mechanics text book. Here, a detailed complete re-derivation of hidden variable equation in David Bohm's 1952 papers is given to assign some physical meanings to Schrodinger's equation. This work shows some required detailed transformation of Schrodinger's Equation into a form of Newtonian second law of motion with definite trajectory with some initial conditions given. All the quantum effects come from the single localized term called the quantum potential.

KEYWORDS: History and Philosophy of Physics, History of Quantum Mechanics, History and Philosophy of Science, David Bohm

INTRODUCTION

Schrodinger's equation is a basic mathematical expression for a single particle in quantum mechanics. In particular, the so-called time dependent Schrodinger's equation has m, mass, ħ, Planck's constant, Φ, the wavefunction, and V, classical potential as follows.

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Phi + V\Phi \qquad (1)$$

Schrodinger's equation has no physical meaning what so ever, according to the standard interpretation of quantum mechanics, the Copenhagen interpretation. Here, a derivation of a Newtonian second form from Schrodinger's equation will be made. This attempt was already made by an American physicist, David Bohm, back in 1952 (Bohm, 1952). Afterwards, many mentioned

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the derivation in a passing statement of their own (Cushing, 1994). Here a full mathematical detail will be given to clarify the derivation in every single step so that any mathematical novice could have a better and deeper understanding of the derivation leading to a Newtonian form.

A detailed mathematical derivation

Let $\Phi = \mathrm{R}e^{is/\hbar}$, with the amplitude $\mathrm{R} = \mathrm{R}(\vec{x}, t)$ and the phase $\mathrm{S} = \mathrm{S}\left(\vec{x}, t\right)$. Then the above equation (1) becomes

$$i\hbar[\frac{\partial \mathrm{R}}{\partial \mathrm{t}} e^{is/\hbar} + \mathrm{R}e^{is/\hbar}\left(\frac{i}{\hbar}\right)\frac{\partial \mathrm{S}}{\partial \mathrm{t}}]$$
$$= -\frac{\hbar^2}{2\mathrm{m}} \nabla \cdot [(\nabla \mathrm{R})e^{is/\hbar} + \mathrm{R}e^{is/\hbar}(\frac{i}{\hbar})\nabla \mathrm{S}] + \mathrm{VR}e^{is/\hbar}$$
$$= -\frac{\hbar^2}{2\mathrm{m}} [(\nabla^2 \mathrm{R})e^{is/\hbar} + (\nabla \mathrm{R})e^{is/\hbar}(\frac{i}{\hbar}) \cdot \nabla \mathrm{S} + (\nabla \mathrm{R})e^{is/\hbar}(\frac{i}{\hbar}) \cdot \nabla \mathrm{S} + \mathrm{R}e^{is/\hbar}\left(\frac{i}{\hbar}\right)^2 (\nabla \mathrm{S})^2 +$$
$$\mathrm{R}e^{is/\hbar}(\frac{i}{\hbar})\nabla^2 \mathrm{S}] + \mathrm{VR}e^{is/\hbar} \qquad (2)$$

Dividing the both sides of the above equation (2) by $e^{is/\hbar}$ gives

$$i\hbar[\frac{\partial \mathrm{R}}{\partial \mathrm{t}} + \mathrm{R}\left(\frac{i}{\hbar}\right)\frac{\partial \mathrm{S}}{\partial \mathrm{t}}]$$
$$= -\frac{\hbar^2}{2\mathrm{m}} [\nabla^2 \mathrm{R} + (\nabla \mathrm{R})(\frac{i}{\hbar}) \cdot \nabla \mathrm{S} + (\nabla \mathrm{R})(\frac{i}{\hbar}) \cdot \nabla \mathrm{S} + \mathrm{R}\left(\frac{i}{\hbar}\right)^2 (\nabla \mathrm{S})^2 + \mathrm{R}(\frac{i}{\hbar})\nabla^2 \mathrm{S}] + \mathrm{VR} \qquad (3)$$

This equation (3) becomes

$$i\hbar\frac{\partial \mathrm{R}}{\partial \mathrm{t}} - \mathrm{R}\frac{\partial \mathrm{S}}{\partial \mathrm{t}} =$$
$$-\frac{\hbar^2}{2\mathrm{m}} [\nabla^2 \mathrm{R} - \left(\frac{R}{\hbar^2}\right)(\nabla \mathrm{S})^2 + i(\nabla \mathrm{R} \cdot \nabla \mathrm{S}\left(\frac{2}{\hbar}\right) + \left(\frac{R}{\hbar}\right)\nabla^2 \mathrm{S})] +$$
$$\mathrm{VR} \qquad (4)$$

The imaginary part of equation (4) gives

$$\hbar\frac{\partial \mathrm{R}}{\partial \mathrm{t}} = -\frac{\hbar^2}{2\mathrm{m}} [\nabla \mathrm{R} \cdot \nabla \mathrm{S}\left(\frac{2}{\hbar}\right) + \left(\frac{R}{\hbar}\right)\nabla^2 \mathrm{S})] \quad (5)$$

This equation (5) simplifies as follows

$$\frac{\partial R}{\partial t} = -\frac{1}{2m}[2\nabla R \cdot \nabla S + R\nabla^2 S)] \qquad (6)$$

The real part of equation (4) gives

$$-R\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m}[\nabla^2 R - \left(\frac{R}{\hbar^2}\right)(\nabla S)^2] + VR \qquad (7)$$

This equation (7) simplifies as follows, if $R \neq 0$

$$\frac{\partial S}{\partial t} = -[-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + \frac{(\nabla S)^2}{2m} + V] \qquad (8)$$

From this equation (8), we can define quantum potential as follows,

$$U \equiv -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} \qquad (9)$$

Now, multiplying equation (6) by 2R gives

$$2R\frac{\partial R}{\partial t} = -\frac{\nabla R \cdot \nabla S}{m}(2R) - \frac{R}{2m}(2R)\nabla^2 S$$

Then, equation (8) becomes as follows,

$$\frac{\partial R^2}{\partial t} = -\frac{\nabla R^2}{m} \cdot \nabla S - \frac{R^2}{m}\nabla^2 S = -\nabla \cdot (R^2\frac{\nabla S}{m}) \qquad (10)$$

This equation (10) then further becomes

$$\frac{\partial R^2}{\partial t} + \nabla \cdot (R^2\frac{\nabla S}{m}) = 0 \qquad (11)$$

, where $R^2 = |\Phi|^2$ since $\Phi = Re^{is/\hbar}$

This equation (11) is the continuity equation of the following standard form.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad (12)$$

Where ρ is density and $\vec{v}$ is velocity.

Comparing the equation (11) and (12) shows $R^2 = (\text{probability})density$ and gives us the following relation

$$\vec{v} = \frac{\nabla S}{m} \qquad (13)$$

Using equation (11), we are able to establish the following momentum with a well defined classical trajectory.

$$\vec{p} = m\vec{v} = \nabla S(\vec{x}, t) \qquad (14)$$

Now, since the total derivative for function f is given by $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$, we have the total derivative of $\vec{p}$ as follows,

$$d\vec{p} = \frac{\partial(\nabla S)}{\partial x} dx + \frac{\partial(\nabla S)}{\partial y} dy + \frac{\partial(\nabla S)}{\partial z} dz + \frac{\partial(\nabla S)}{\partial t} dt$$

Therefore,

$$\begin{aligned} \frac{d\vec{p}}{dt} &= \frac{\partial(\nabla S)}{\partial x}\frac{dx}{dt} + \frac{\partial(\nabla S)}{\partial y}\frac{dy}{dt} + \frac{\partial(\nabla S)}{\partial z}\frac{dz}{dt} + \frac{\partial(\nabla S)}{\partial t} \\ &= \frac{1}{m}\left[m\frac{dx}{dt}\frac{\partial(\nabla S)}{\partial x} + m\frac{dy}{dt}\frac{\partial(\nabla S)}{\partial y} + m\frac{dz}{dt}\frac{\partial(\nabla S)}{\partial z}\right] + \frac{\partial(\nabla S)}{\partial t} \\ &= \frac{1}{m}\left[m\frac{dx}{dt}\frac{\partial}{\partial x} + m\frac{dy}{dt}\frac{\partial}{\partial y} + m\frac{dz}{dt}\frac{\partial}{\partial z}\right](\nabla S) + \frac{\partial(\nabla S)}{\partial t} = \frac{1}{m}[m\vec{v} \cdot \nabla](\nabla S) + \frac{\partial(\nabla S)}{\partial t} \\ &= \frac{1}{m}[\vec{p} \cdot \nabla](\nabla S) + \nabla\frac{\partial S}{\partial t} = \frac{1}{m}[\nabla S \cdot \nabla](\nabla S) + \nabla(\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} - \frac{(\nabla S)^2}{2m} - V]) \end{aligned} \qquad (15)$$

In the above equation (15), we used equation (14) and (8)

from the vector calculus (Griffiths, 2017), we can recall

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$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B})+(\vec{A}\cdot\nabla)\vec{B}+\vec{B}\times(\nabla\times\vec{A})+(\vec{B}\cdot\nabla)\vec{A} \quad (16)$$

The first term in equation (15) corresponds to the second term of the above relation (16) when $\vec{A} = \vec{B} = \nabla S$. This comparison gives

$$\nabla(\nabla S)^2 = \nabla(\nabla S\cdot\nabla S) = \nabla S\times(\nabla\times\nabla S)+(\nabla S\cdot\nabla)\nabla S+\nabla S\times(\nabla\times\nabla S)+(\nabla S\cdot\nabla)\nabla S = 0 + 2(\nabla S\cdot\nabla)\nabla S + 0 \quad (17)$$

Now, equation (15) becomes

$$\frac{d\vec{p}}{dt} = \frac{1}{2m}\nabla(\nabla S)^2 + \nabla(\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} - \frac{(\nabla S)^2}{2m} - V]) \quad (18)$$

However, since $\nabla(\nabla S)^2 = 0 = 2(\nabla S\cdot\nabla)\nabla S$ from equation (17), the first and the third term of equation (18) vanish

Therefore, using equation (9) for U, the quantum potential.

$$\frac{d\vec{p}}{dt} = \nabla(\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} - V) = -\nabla(U+V) \equiv F = ma \quad (19)$$

Equation (19) is now a form of Newtonian second law of motion with definite trajectory with some initial conditions given. All the quantum effects come from the single localized term called the quantum potential given by equation (9).

CONCLUSION

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Phi + V\Phi$$

The above well-known time dependent Schrodinger's equation with m, mass, ħ, Planck's constant, Φ, the wavefunction, and V, classical potential can be transferred to a new Newtoniam form of the following

$$\frac{d\vec{p}}{dt} = \nabla(\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} - V)$$

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